1. Problem Set 1 has been posted
2. It is due on Wednesday September 27
Today’s Plan

1. Overview
2. Log-supermodularity
3. R-R model
4. Cross-sectional predictions
5. Comparative static predictions
1. Overview
Assignment Models in the Trade Literature

- Small but rapidly growing literature using assignment models in an international context:

- **What do these models have in common?**
  - Factor allocation can be summarized by an assignment function
  - Large number of factors and/or goods

- **What is the main difference between these models?**
  - *Matching*: Two sides of each match in finite supply (as in Becker 1973)
  - *Sorting*: One side of each match in infinite supply (as in Roy 1951)
I will restrict myself to sorting models, e.g. Ohnsorge and Trefler (2007), Costinot (2009), and Costinot and Vogel (2010)

- Production functions are linear, as in Ricardian model
- But more than one factor per country, as in Roy model
- **Ricardo-Roy model**

**Objectives:**

1. Describe how these models relate to “standard” neoclassical models
2. Introduce simple tools from the mathematics of complementarity
3. Use tools to derive cross-sectional and comparative static predictions

This is very much a methodological lecture. If you are interested in more specific applications, read the papers...
2. Log-Supermodularity
Definition 1 A function $g: X \rightarrow \mathbb{R}^+$ is log-supermodular if for all $x, x' \in X$, $g(\max(x, x')) \cdot g(\min(x, x')) \geq g(x) \cdot g(x')$

Bivariate example:

- If $g : X_1 \times X_2 \rightarrow \mathbb{R}^+$ is log-spm, then $x'_1 \geq x''_1$ and $x'_2 \geq x''_2$ imply
  
  $$g(x'_1, x'_2) \cdot g(x''_1, x''_2) \geq g(x'_1, x''_2) \cdot g(x'_1, x''_2),$$

- If $g$ is strictly positive, this can be rearranged as
  
  $$g(x'_1, x'_2) / g(x''_1, x''_2) \geq g(x'_1, x''_2) / g(x''_1, x''_2).$$
Log-supermodularity

Results

**Lemma 1.** \( g, h : X \to \mathbb{R}^+ \) log-spm \( \Rightarrow \) gh log-spm

**Lemma 2.** \( g : X \to \mathbb{R}^+ \) log-spm \( \Rightarrow \) \( G(x_{-i}) = \int_{x_i} g(x) \, dx_i \) log-spm

**Lemma 3.** \( g : T \times X \to \mathbb{R}^+ \) log-spm \( \Rightarrow \)

\[ x^*(t) \equiv \arg \max_{x \in X} g(t, x) \text{ increasing in } t \]
3. R-R Model
Consider a world economy with:

1. Multiple countries with characteristics $\gamma \in \Gamma$
2. Multiple goods or sectors with characteristics $\sigma \in \Sigma$
3. Multiple factors of production with characteristics $\omega \in \Omega$

Factors are immobile across countries, perfectly mobile across sectors.
Goods are freely traded at world price $p(\sigma) > 0$
Technology

- Within each sector, factors of production are perfect substitutes

\[ Q(\sigma, \gamma) = \int_{\Omega} A(\omega, \sigma, \gamma) L(\omega, \sigma, \gamma) d\omega, \]

- \( A(\omega, \sigma, \gamma) \geq 0 \) is productivity of \( \omega \)-factor in \( \sigma \)-sector and \( \gamma \)-country

- **A1** \( A(\omega, \sigma, \gamma) \) is log-supermodular

- A1 implies, in particular, that:
  1. High-\( \gamma \) countries have a comparative advantage in high-\( \sigma \) sectors
  2. High-\( \omega \) factors have a comparative advantage in high-\( \sigma \) sectors
Factor Endowments

- $V(\omega, \gamma) \geq 0$ is inelastic supply of $\omega$-factor in $\gamma$-country
- **A2** $V(\omega, \gamma)$ is log-supermodular

A2 implies that:
- High-$\gamma$ countries are relatively more abundant in high-$\omega$ factors
- Preferences will be described later on when we do comparative statics
4. Cross-Sectional Predictions
4.1 Competitive Equilibrium

- We take the price schedule \( p(\sigma) \) as given [small open economy]
- In a competitive equilibrium, \( L \) and \( w \) must be such that:
  1. Firms maximize profit
     \[
     p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) \leq 0, \text{ for all } \omega \in \Omega \\
     p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) = 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0
     \]
  2. Factor markets clear
     \[
     V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma) d\sigma, \text{ for all } \omega \in \Omega
     \]
4.2 Patterns of Specialization

Predictions

- Let $\Sigma(\omega, \gamma) \equiv \{\sigma \in \Sigma | L(\omega, \sigma, \gamma) > 0\}$ be the set of sectors in which factor $\omega$ is employed in country $\gamma$.

**Theorem [PAM]** $\Sigma(\cdot, \cdot)$ is increasing.

**Proof:**

1. Profit maximization $\Rightarrow \Sigma(\omega, \gamma) = \arg \max_{\sigma \in \Sigma} p(\sigma) A(\omega, \sigma, \gamma)$
2. A1 $\Rightarrow p(\sigma) A(\omega, \sigma, \gamma)$ log-spm by Lemma 1
3. $p(\sigma) A(\omega, \sigma, \gamma)$ log-spm $\Rightarrow \Sigma(\cdot, \cdot)$ increasing by Lemma 3

**Corollary** High-$\omega$ factors specialize in high-$\sigma$ sectors.

**Corollary** High-$\gamma$ countries specialize in high-$\sigma$ sectors.
4.2 Patterns of Specialization
Relation to the Ricardian literature

- Ricardian model ≡ Special case w/ $A(\omega, \sigma, \gamma) \equiv A(\sigma, \gamma)$
- Previous corollary can help explain:

1. **Multi-country-multi-sector Ricardian model;** Jones (1961)
   - According to Jones (1961), efficient assignment of countries to goods solves $\max \sum \ln A(\sigma, \gamma)$
   - According to Corollary, $A(\sigma, \gamma)$ log-spm implies PAM of countries to goods; Becker (1973), Kremer (1993), Legros and Newman (1996).

   - Papers vary in terms of source of “institutional dependence” $\sigma$ and ”institutional quality” $\gamma$
   - ...but same fundamental objective: providing micro-theoretical foundations for the log-supermodularity of $A(\sigma, \gamma)$
Previous results are about the set of goods that each country produces.

**Question:** Can we say something about how much each country produces? Or how much it employs in each particular sector?

**Answer:** Without further assumptions, the answer is no.
4.3 Aggregate Output, Revenues, and Employment

Additional assumptions

- **A3.** The profit-maximizing allocation $L$ is unique
- **A4.** Factor productivity satisfies $A(\omega, \sigma, \gamma) \equiv A(\omega, \sigma)$

**Comments:**

1. A3 requires $p(\sigma) A(\omega, \sigma, \gamma)$ to be maximized in a *single* sector
2. A3 is an implicit restriction on the demand-side of the world-economy
   - ... but it becomes milder and milder as the number of factors or countries increases
   - ... generically true if continuum of factors
3. A4 implies no Ricardian sources of CA across countries
   - Pure Ricardian case can be studied in a similar fashion
   - Having multiple sources of CA is more complex (Costinot 2009)
4.3 Aggregate Output, Revenues, and Employment

Output predictions

- **Theorem**: If A3 and 4 hold, then $Q(\sigma, \gamma)$ is log-spm.

- **Proof**:
  1. Let $\Omega(\sigma) \equiv \{ \omega \in \Omega | p(\sigma) A(\omega, \sigma) > \max_{\sigma' \neq \sigma} p(\sigma') A(\omega, \sigma') \}$. A3 and A4 imply $Q(\sigma, \gamma) = \int 1_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma) V(\omega, \gamma) d\omega$
  2. $A1 \Rightarrow \tilde{A}(\omega, \sigma) \equiv 1_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma)$ log-spm
  3. $A2$ and $\tilde{A}(\omega, \sigma)$ log-spm $+$ Lemma 1 $\Rightarrow \tilde{A}(\omega, \sigma) V(\omega, \gamma)$ log-spm
  4. $\tilde{A}(\omega, \sigma) V(\omega, \gamma)$ log-spm $+$ Lemma 2 $\Rightarrow Q(\sigma, \gamma)$ log-spm

- **Intuition**:
  1. $A1 \Rightarrow$ high $\omega$-factors are assigned to high $\sigma$-sectors
  2. $A2 \Rightarrow$ high $\omega$-factors are more likely in high $\gamma$-countries
Corollary. Suppose that A3 and A4 hold. If two countries produce $J$ goods, with $\gamma_1 \geq \gamma_2$ and $\sigma_1 \geq ... \geq \sigma_J$, then the high-$\gamma$ country tends to specialize in the high-$\sigma$ sectors:

$$\frac{Q(\sigma_1, \gamma_1)}{Q(\sigma_1, \gamma_2)} \geq ... \geq \frac{Q(\sigma_J, \gamma_1)}{Q(\sigma_J, \gamma_2)}$$
4.3 Aggregate Output, Revenues, and Employment

Employment and revenue predictions

- Let \( L(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} V(\omega, \gamma) d\omega \) be aggregate employment
- Let \( R(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} r(\omega, \sigma) V(\omega, \gamma) d\omega \) be aggregate revenues

**Corollary.** Suppose that A3 and A4 hold. If two countries produce \( J \) goods, with \( \gamma_1 \geq \gamma_2 \) and \( \sigma_1 \geq \ldots \geq \sigma_J \), then aggregate employment and aggregate revenues follow the same pattern as aggregate output:

\[
\frac{L(\sigma_1, \gamma_1)}{L(\sigma_1, \gamma_2)} \geq \ldots \geq \frac{L(\sigma_J, \gamma_1)}{L(\sigma_J, \gamma_2)} \quad \text{and} \quad \frac{R(\sigma_1, \gamma_1)}{R(\sigma_1, \gamma_2)} \geq \ldots \geq \frac{R(\sigma_J, \gamma_1)}{R(\sigma_J, \gamma_2)}
\]
4.3 Aggregate Output, Revenues, and Employment

Relation to the previous literature

1. **Worker Heterogeneity and Trade**
   - Generalization of Ruffin (1988):
     - Continuum of factors, Hicks-neutral technological differences
     - Results hold for an arbitrarily large number of goods and factors
   - Generalization of Ohnsorge and Trefler (2007):
     - No functional form assumption (log-normal distribution of human capital, exponential factor productivity)

2. **Firm Heterogeneity and Trade**
   - “Factors” ≡ “Firms” with productivity $\omega$
   - “Countries” ≡ “Industries” with characteristic $\gamma$
   - “Sectors” ≡ “Organizations” with characteristic $\sigma$
   - $Q(\sigma, \gamma)$ ≡ Sales by firms with ”$\sigma$-organization” in “$\gamma$-industry”
   - In previous papers, $f(\omega, \gamma)$ log-spm is crucial, Pareto is not
5. Comparative Static Predictions
5.1 Closing The Model

Additional assumptions

- Assumptions A1-4 are maintained
- In order to do comparative statics, we also need to specify the demand side of our model:

\[
U = \left\{ \int_{\sigma \in \Sigma} \left[ C(\sigma, \gamma) \right]^\frac{\varepsilon-1}{\varepsilon} d\sigma \right\}^{\frac{\varepsilon}{\varepsilon-1}}
\]

- For expositional purposes, we will also assume that:
  - \( A(\omega, \sigma) \) is strictly log-supermodular
  - Continuum of factors and sectors: \( \Sigma \equiv [\sigma, \bar{\sigma}] \) and \( \Omega \equiv [\omega, \bar{\omega}] \)
5.1 Closing the Model

Autarky equilibrium

Autarky equilibrium is a set of functions \((Q, C, L, p, w)\) such that:

1. **Firms maximize profit**

\[
p(\sigma)A(\omega, \sigma) - w(\omega, \gamma) \leq 0, \text{ for all } \omega \in \Omega
\]

\[
p(\sigma)A(\omega, \sigma) - w(\omega, \gamma) = 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0
\]

2. **Factor markets clear**

\[
V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma) d\sigma, \text{ for all } \omega \in \Omega
\]

3. **Consumers maximize their utility and good markets clear**

\[
C(\sigma, \gamma) = I(\gamma) \times p(\sigma)^{-\varepsilon} = Q(\sigma, \gamma)
\]
5.1 Closing the Model

Properties of autarky equilibrium

- **Lemma 1** *In autarky equilibrium, there exists an increasing bijection* $M : \Omega \to \Sigma$ *such that* $L(\omega, \sigma) > 0$ *if and only if* $M(\omega) = \sigma$.

- **Lemma 2** *In autarky equilibrium, $M$ and $w$ satisfy*

  $\frac{dM(\omega, \gamma)}{d\omega} = A[\omega, M(\omega, \gamma)] V(\omega, \gamma) \frac{l(\gamma) \times \{p[M(\omega), \gamma]\}}{-\varepsilon}$ (1)

  $\frac{d\ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega)]}{\partial \omega}$ (2)

  *with* $M(\omega, \gamma) = \sigma$, $M(\overline{\omega}, \gamma) = \overline{\sigma}$, *and* $p[M(\omega, \gamma), \gamma] = w(\omega, \gamma) / A[\omega, M(\omega, \gamma)]$. 
5.1 Closing the Model
Properties of autarky equilibrium

- **Proof of Lemma 1**: Similar to proof of PAM in 4.2

- **Proof of Lemma 2**:
  1. Profit-maximization implies
     \[
     \ln w(\omega, \gamma) = \max_\sigma \{ \ln p(\sigma) + \ln A(\omega, \sigma) \}
     \]
  2. Thus envelope theorem gives
     \[
     \frac{d \ln w(\omega, \gamma)}{d \omega} = \frac{\partial \ln A[\omega, M(\omega)]}{\partial \omega}
     \]
  3. Factor market + good market clearing imply
     \[
     \int_{\sigma}^{M(\omega, \gamma)} I(\gamma) \times p(\sigma)^{-\varepsilon} A(\sigma, \gamma) d\sigma = \int_{\omega} V(v, \gamma) dv
     \]
  4. Differentiating with respect to \( \omega \) gives (1)
5.2 Changes in Factor Supply

**Question:** What happens if we change country characteristics from $\gamma$ to $\gamma' \leq \gamma$?

If $\omega$ is worker “skill”, this can be thought of as a change in terms of “skill abundance”:

$$\frac{V(\omega, \gamma)}{V(\omega', \gamma)} \geq \frac{V(\omega, \gamma')}{V(\omega', \gamma')}, \text{ for all } \omega > \omega'$$

If $V(\omega, \gamma)$ was a normal distribution, this would correspond to a change in the mean.
5.2 Changes in Factor Supply

Consequence for factor allocation

- **Lemma** \( M(\omega, \gamma') \geq M(\omega, \gamma) \) for all \( \omega \in \Omega \)

- **Intuition:**
  - If there are relatively more low-\( \omega \) factors, more sectors should use them
  - From a sector standpoint, this requires *factor downgrading*
Proof: If there is $\omega$ s.t. $M(\omega, \gamma') < M(\omega, \gamma)$, then there exist:

1. $M(\omega_1, \gamma') = M(\omega_1, \gamma) = \sigma_1$, $M(\omega_2, \gamma') = M(\omega_2, \gamma) = \sigma_2$, and $\frac{M_\omega(\omega_1, \gamma')}{M_\omega(\omega_2, \gamma')} \leq \frac{M_\omega(\omega_1, \gamma)}{M_\omega(\omega_2, \gamma)}$

2. Equation (1) $\implies \frac{V(\omega_2, \gamma')}{V(\omega_1, \gamma')} \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} \geq \frac{V(\omega_2, \gamma)}{V(\omega_1, \gamma)} \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)}$

3. $V \ log-sp \implies \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} \geq \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)}$

4. Equation (2) $+ zero \ profits \implies \frac{d \ln p(\sigma, \gamma)}{d \sigma} = -\frac{\partial \ln A[M^{-1}(\sigma, \gamma), \sigma]}{\partial \sigma}$

5. $M^{-1}(\sigma, \gamma) < M^{-1}(\sigma, \gamma')$ for $\sigma \in (\sigma_1, \sigma_2) + A \ log-sp \implies \frac{p(\sigma_1, \gamma)}{p(\sigma_2, \gamma)} < \frac{p(\sigma_1, \gamma')}{p'(\sigma_2, \gamma')}$

6. $\frac{p(\sigma_1, \gamma)}{p(\sigma_2, \gamma)} < \frac{p(\sigma_1, \gamma')}{p'(\sigma_2, \gamma')} + CES \implies \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} < \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)}$. A contradiction
5.2 Changes in Factor Supply

Consequence for factor prices

- A decrease from $\gamma$ to $\gamma'$ implies *pervasive rise in inequality*:

\[
\frac{w(\omega, \gamma')}{w(\omega', \gamma')} \geq \frac{w(\omega, \gamma)}{w(\omega', \gamma)}, \text{ for all } \omega > \omega'\]

- The mechanism is simple:

  1. Profit-maximization implies

\[
\frac{d \ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma)]}{\partial \omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma')]}{\partial \omega}
\]

  2. Since $A$ is log-supermodular, task upgrading implies

\[
\frac{d \ln w(\omega, \gamma')}{d\omega} \geq \frac{d \ln w(\omega, \gamma)}{d\omega}
\]
In Costinot Vogel (2010), we also consider changes in diversity.

This corresponds to the case where there exists $\hat{\omega}$ such that $V(\omega, \gamma)$ is log-supermodular for $\omega > \hat{\omega}$, but log-submodular for $\omega < \hat{\omega}$.

We also consider changes in factor demand (Computerization?):

\[
U = \left\{ \int_{\sigma \in \Sigma} B(\sigma, \gamma) \left[ C(\sigma, \gamma) \right]^{\frac{\epsilon-1}{\epsilon}} d\sigma \right\}^{\frac{\epsilon}{\epsilon-1}}
\]
5.3 North-South Trade

Free trade equilibrium

- Two countries, Home ($H$) and Foreign ($F$), with $\gamma_H \geq \gamma_F$
- A competitive equilibrium in the world economy under free trade is s.t.

\[
\frac{dM(\omega, \gamma_T)}{d\omega} = \frac{A[\omega, M(\omega, \gamma_T)] V(\omega, \gamma_T)}{l_T \times \{p[M(\omega, \gamma_T), \gamma_T]\}^{-\epsilon}},
\]

\[
\frac{d \ln w(\omega, \gamma_T)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma_T)]}{\partial \omega},
\]

where:

\[
M(\omega, \gamma_T) = \sigma \text{ and } M(\bar{\omega}, \gamma_T) = \bar{\sigma}
\]

\[
p[M(\omega, \gamma_T), \gamma_T] = w(\omega, \gamma_T) A[\omega, M(\omega, \gamma_T)]
\]

\[
V(\omega, \gamma_T) \equiv V(\omega, \gamma_H) + V(\omega, \gamma_F)
\]
5.3 North South Trade

Free trade equilibrium

- **Key observation:**
  \[
  \frac{V(\omega, \gamma_H)}{V(\omega', \gamma_H)} \geq \frac{V(\omega, \gamma_F)}{V(\omega', \gamma_F)}, \text{ for all } \omega > \omega' \Rightarrow \frac{V(\omega, \gamma_H)}{V(\omega', \gamma_H)} \geq \frac{V(\omega, \gamma_T)}{V(\omega', \gamma_T)} \geq \frac{V(\omega, \gamma_F)}{V(\omega', \gamma_F)}
  \]

- Continent-by-continuum extensions of two-by-two HO results:
  
  1. **Changes in skill-intensities:**
     \[
     M(\omega, \gamma_H) \leq M(\omega, \gamma_T) \leq M(\omega, \gamma_F), \text{ for all } \omega
     \]
  
  2. **Strong Stolper-Samuelson effect:**
     \[
     \frac{w(\omega, \gamma_H)}{w(\omega', \gamma_H)} \leq \frac{w(\omega, \gamma_T)}{w(\omega', \gamma_T)} \leq \frac{w(\omega, \gamma_F)}{w(\omega', \gamma_F)}, \text{ for all } \omega > \omega'
     \]
5.3 North South Trade
Other Predictions

- North-South trade driven by factor demand differences:
  - Same logic gets to the exact opposite results
  - Correlation between factor demand and factor supply considerations matters

- One can also extend analysis to study “North-North” trade:
  - It predicts wage polarization in the more diverse country and wage convergence in the other
Costinot and Vogel (2015, ARE) review a number of extensions:

1. Monopolistic competition (Sampson 2014, AEJ)
2. Vertical specialization (Costinot, Vogel and Wang 2013, RES)
3. Heterogeneous preferences (Redding 2013)
4. Endogenous skills (Blanchard and Willman 2013)
What’s next?

Theory:
- Learning by doing (build on GRH 2010?)
- Labor market frictions (build on Teulings 2003?)
- Endogenous technology adoption

Empirics:
- Revisiting the consequences of trade liberalization (Adao 2016)
- Parametric applications with extreme value distributions?
- More flexible approaches?