Plan for Today’s Lecture

1. Allen and Arkolakis (2014)
   1. Introduction
   2. Model set-up
   3. Equilibrium characterization
   4. Estimation
   5. Counterfactuals
Allen and Arkolakis (QJE, 2014)

- General spatial economic model
  - Combines gravity structure with labor mobility.
  - Any continuous bilateral trade costs ("geographic location").
  - Any continuous topography of amenities and productivities ("local characteristics").

- Flexible productivity and amenity spillovers
  - Special cases are isomorphic to seminal economic geography models.

- Tractable general equilibrium structure
  - Show solutions are special cases of well-understood mathematical systems.
  - Characterize the conditions for existence, uniqueness, and stability.
  - Derive simple equations governing relationship between equilibrium economic activity and geography.
Geography

Compact set $S$ of locations inhabited by $\bar{L}$ workers.

Location $i \in S$ is endowed with:
- Differentiated variety (Armington assumption).
- Productivity $\bar{A}(i)$.
- Amenity $\bar{u}(i)$.

For all $i, j \in S$, let the iceberg bilateral trade cost be $T(i, j)$.

Terminology
- $\bar{A}$ and $\bar{u}$ are the local characteristics.
- $T$ determines geographic location.
- Together, $\bar{A}$, $\bar{u}$, and $T$ comprise the geography of $S$.

A geography is regular if $\bar{A}$, $\bar{u}$ and $T$ are continuous and bounded above and below by strictly positive numbers.
Endowed with identical CES preferences over differentiated varieties with elasticity of substitution $\sigma > 1$.

Can choose to live/work in any location $i \in S$.

Receive wage $w(i)$ for their inelastically supplied unit of labor.

Welfare in location $i$ is:

$$W(i) = \left( \int_{s \in S} q(s, i) \frac{\sigma - 1}{\sigma} \, ds \right)^{\frac{\sigma}{\sigma - 1}} u(i)$$

where $q(s, i)$ is the per capita quantity consumed in location $i$ of the good produced in location $s$ and $u(i)$ is the local amenity.
Labor is the only factor of production, $L(i)$ is the density of workers.

Productivity of worker in location $i$ is $A(i)$.

Perfect competition implies price of good from $i$ is $\frac{w(i)}{A(i)} T(i, j)$ in location $j$.

Functions $w$ and $L$ comprise the distribution of economic activity.
Productivity and amenity spillovers

Productivity is potentially subject to externalities:

\[ A(i) = \bar{A}(i) L(i)^{\alpha} \]

Amenities are potentially subject to externalities:

\[ u(i) = \bar{u}(i) L(i)^{\beta} \]

Isomorphisms:

Monopolistic competition with free entry: \( \alpha = \frac{1}{\sigma-1} \).

Cobb-Douglas preferences over non-tradable sector: \( \beta = -\frac{1-\gamma}{\gamma} \).

Heterogeneous (extreme-value) worker preferences: \( \beta = -\frac{1}{\theta} \).
Markets are said to **clear** if for all $i \in S$:

$$w(i) L(i) = \int_S X(i, s) \, ds,$$

where $X(i, j)$ is the value of trade flows from $i \in S$ to $j \in S$.

Welfare is said to be **equalized** if there exists $W \in \mathbb{R}_{++}$ such that for all $i \in S$, $W(i) \leq W$, with the equality strict if $L(i) > 0$. 
A spatial equilibrium is a distribution of economic activity such that:
Markets clear,
Welfare is equalized,
The aggregate labor market clears, i.e. \[ \int_S L(s) \, ds = \bar{L}. \]
Characterization

A spatial equilibrium is regular if \( L \) and \( w \) are strictly positive and continuous.

A spatial equilibrium is point-wise locally stable if \[ \frac{dW(i)}{dL(i)} < 0 \] for all \( i \in S \).
Suppose $\alpha = \beta = 0$ so that $A(i) = \bar{A}(i)$ and $u(i) = \bar{u}(i)$.

From welfare equalization:

$$w(i)^{1-\sigma} = W^{1-\sigma} \int_S T(s, i)^{1-\sigma} u(i)^{\sigma-1} A(s)^{\sigma-1} w(s)^{1-\sigma} ds$$

From balanced trade:

$$L(i) w(i)^{\sigma} = W^{1-\sigma} \int_S T(i, s)^{1-\sigma} A(i)^{\sigma-1} u(s)^{\sigma-1} L(s) w(s)^{\sigma} ds$$

These two equations are eigenfunctions of $w(i)^{1-\sigma}$ and $L(i) w(i)^{\sigma}$, respectively.
Equilibrium without spillovers: Theorem

**Theorem**

For any regular geography with exogenous productivity and amenities:

1. There exists a unique equilibrium.

2. The equilibrium is regular and point-wise locally stable.

3. Equilibrium can be determined using an iterative procedure.

**Proof.**

Application of Jentzsch’s theorem (generalization of the Perron-Frobenius theorem) and Fredholm’s Theorems.
Equilibrium with spillovers

Can rewrite balanced trade and utility equalization as:

\[ L(i)^{1-\alpha(\sigma-1)} w(i)^\sigma = W^{1-\sigma} \int_S T(i, s)^{1-\sigma} \tilde{A}(i)^{\sigma-1} \tilde{u}(s)^{\sigma-1} L(s)^{1+\beta(\sigma-1)} w(s)^\sigma \, ds \]

\[ w(i)^{1-\sigma} L(i)^{\beta(1-\sigma)} = W^{1-\sigma} \int_S T(s, i)^{1-\sigma} \tilde{A}(s)^{\sigma-1} \tilde{u}(i)^{\sigma-1} w(s)^{1-\sigma} L(s)^{\alpha(\sigma-1)} \, ds \]

If \( T(i, s) = T(s, i) \) for all \( i, s \in S \) then the solution can be written as:

\[ A(i)^{\sigma-1} w(i)^{1-\sigma} = \phi L(i) w(i)^\sigma u(i)^{\sigma-1} \]

\[ L(i)^{\tilde{\sigma}\gamma_1} = K_1(i) W^{1-\sigma} \int_S T(s, i)^{1-\sigma} K_2(s) \left( L(s)^{\tilde{\sigma}\gamma_1} \right)^{\gamma_2} \gamma_1 \, ds, \]

where \( K_1(i) \) and \( K_2(i) \) are functions of \( \tilde{A}(i) \) and \( \tilde{u}(i) \), \( \gamma_1, \gamma_2 \), and \( \tilde{\sigma} \) are functions of \( \alpha, \beta, \) and \( \sigma \).

The last equation is a Hammerstein non-linear integral equation.
Theorem

Consider any regular geography with endogenous productivity and amenities with $T$ symmetric. Define $\gamma_1 \equiv 1 - \alpha (\sigma - 1) - \beta \sigma$ and $\gamma_2 \equiv 1 + \alpha \sigma + (\sigma - 1) \beta$. If $\gamma_1 \neq 0$, then:

1. There exists a regular equilibrium.
2. If $\gamma_1 < 0$, no regular equilibria are point-wise locally stable.
3. If $\gamma_1 > 0$, all equilibria are regular and point-wise locally stable.
4. If $\frac{\gamma_2}{\gamma_1} \in [-1, 1]$, the equilibrium is unique.
5. If $\frac{\gamma_2}{\gamma_1} \in (-1, 1]$, the equilibrium can be determined using an iterative procedure.
Result 5 implies this can be very easy to do...
Figure: Equilibria with endogenous amenities and productivity

Existence, Stability, Uniqueness
\[ \alpha + \beta \leq 0 \]

Existence
\[ \gamma_1 = 1 - (\sigma - 1)\alpha - \beta \sigma < 0 \]

Existence, Stability
\[ \gamma_1 = 1 - (\sigma - 1)\alpha - \beta \sigma > 0 \]

"Black-hole"
\[ \gamma_1 = 1 - (\sigma - 1)\alpha - \beta \sigma = 0 \]
Geography and the equilibrium distribution of labor

When trade costs are symmetric, equilibrium distribution of labor can be written as a log-linear function of the underlying geography:

$$\gamma_1 \ln L(i) = C_L + (\sigma - 1) \ln \bar{A}(i) + \sigma \ln \bar{u}(i) + (1 - 2\sigma) \ln P(i)$$

Implications:

When equilibrium is point-wise locally stable, population is increasing in $\bar{A}$ and $\bar{u}$.

Price index is a sufficient statistic for geographic location.

Conditional on price index, productivity and amenity spillovers only affect elasticity of $L(i)$ to geography.
Geographic trade costs

Suppose $S$ is a compact surface (e.g. a line, plane, or sphere).

Let $\tau : S \rightarrow \mathbb{R}_+$ be a continuous function, where $\tau (i)$ is the instantaneous trade cost of traveling over location $i \in S$.

Define the **geographic trade cost** $T(i, j) = f(t(i, j))$, $f' > 0$, $f(0) = 1$ to be the total iceberg trade cost incurred traveling along the least cost route from $i$ to $j$, i.e.

$$
 t(i, j) = \min_{\gamma \in \Gamma(i, j)} \int_0^1 \tau(\gamma(t)) \| \frac{d\gamma(t)}{dt} \| dt
$$

where $\gamma : [0, 1] \rightarrow S$ is a path and $\Gamma(i, j) \equiv \{ \gamma \in C^1 | \gamma(0) = i, \gamma(1) = j \}$ is the set of all paths.

$f(t) = \exp(t)$ natural choice since $\prod_{0}^{1} (1 + \tau(x) \, dx) = \exp \left( \int_{0}^{1} \tau(x) \, dx \right)$, but can show $T$ satisfied triangle inequality $\iff f$ is log subadditive.
Equation (1) appears in a number of branches of physics. A necessary condition for its solution is the following *eikonal* equation:

$$||\nabla t(i,j)|| = \tau(j)$$

where the gradient is taken with respect to $j$.

Simple geometric interpretation: the trade cost contour expands outward in the direction orthogonal to the contour at a rate inversely proportional to the instantaneous trade cost.
Eikonal Equation Illustrated

\[ \{ j \mid t(i, j) = C \} \]
Eikonal Equation Illustrated

\{j \mid t(i,j) = C\}
Eikonal Equation Illustrated

\{j' \mid t(i,j') = C + \epsilon\}

\{j \mid t(i,j) = C\}
Eikonal Equation Illustrated
Eikonal Equation Illustrated
Trade and the topography of the spatial economy: Overview

Estimate the geography of the United States.

Estimate bilateral trade costs

Given trade costs, identify (composite) productivities and amenities

Quantify the importance of geographic location.

Perform counterfactual exercise: remove the Interstate Highway System. Note: Cannot identify $\sigma$, $\alpha$ or $\beta$; they do analysis for a large variety of $\alpha$ and $\beta$, assume $\sigma = 9$. 
Estimating bilateral trade costs

Goal: Find trade costs that best rationalize the bilateral trade flows observed in 2007 Commodity Flow Survey (CFS).

Three step process: Using **Fast Marching Method** (which operationalizes the Eikonal equation) and observed **transportation network**, calculate the (normalized) distance between every CFS area for each major mode of travel (road, rail, air, and water).

Using a **discrete choice** framework and observed **mode-specific bilateral trade shares**, estimate the relative cost of each mode of travel.

Using a **gravity** model and observed **total bilateral trade flows**, pin down normalization (and incorporate non-geographic trade costs).
Estimating trade costs

For any $i, j \in S$, suppose $\exists$ traders $t \in T$ choosing mode $m \in \{1, \ldots, M\}$ of transit where cost is:

$$\exp (\tau_m d_m (i, j) + f_m + \nu_{tm})$$

Then mode-specific bilateral trade shares are:

$$\pi_m (i, j) = \frac{\exp (-a_m d_m (i, j) - b_m)}{\sum_k (\exp (-a_k d_k (i, j) - b_k))},$$

where $a_m \equiv \theta \tau_m$ and $b_m \equiv \theta f_m$.

Combined with model, yields gravity equation:

$$\ln X_{ij} = \frac{\sigma - 1}{\theta} \ln \sum_m (\exp (-a_m d_{mij} - b_m)) + (1 - \sigma) \beta' \ln C_{ij} + \delta_i + \delta_j$$

Estimate $a_m$ and $b_m$ using bilateral trade shares, $\theta$ using gravity equation.

Notes:

No mode switching.

Assume $f_{road} = 0$ to pin down scale.
Estimating $A$ and $u$

Can we identify a topography of productivities $A$ and amenities $u$ consistent with the estimated $T$ and observed distribution of economic activity ($w$ and $L$)?

Yes (see Theorem 3 in the paper).

Intuition: consider locations $a$ and $b$ with identical bilateral trade costs, i.e., for all $s \in S$, $T(a,s) = T(b,s)$. Then:

Utility equalization implies $\frac{u(b)}{u(a)} = \frac{w(a)}{w(b)}$.

Balanced trade implies $\frac{A(a)}{A(b)} = \left( \frac{L(a)w(a)^\sigma}{L(b)w(b)^\sigma} \right)^{\frac{1}{\sigma-1}}$.

Note: $\bar{A}$ and $\bar{u}$ cannot be identified without knowledge of $\alpha$ and $\beta$. 
Figure 12: United States population density and wages in 2000

Notes: This figure shows the relative population density (top) and wages (bottom) within the United States in the year 2000 by decile. The data are reported at the county level; red (blue) indicate higher (lower) deciles. (Source: MPC (2011a)).
Figure 13: Estimated composite productivity and amenity

Composite productivity

Composite amenity

Notes: This figure shows the estimated composite productivity (top) and amenity (bottom) by decile. The data are reported at the county level; red (blue) indicate higher (lower) deciles.
Importance of geographic location

What explains the difference in economic activity across space?

Model yields following equilibrium relationship:

$$\ln Y(i) = C + \gamma_1 \ln \bar{A}(i) + \gamma_2 \ln \bar{u}(i) + \gamma_3 \ln P(i),$$

where $Y(i) \equiv w(i)L(i)$.

Apply a Shapley decomposition to determine what fraction of the variation in $\ln Y(i)$ is due to local characteristics (i.e. $\ln \bar{A}(i)$ and $\ln \bar{u}(i)$) and geographic location (i.e. $\ln P(i)$).

Do for all $\alpha \in [0, 1], \beta \in [-1, 0]$ for robustness.
Removing the IHS: Cost-benefit analysis

Estimated annual cost of the IHS (interstate highway system): \( \approx $100 \) billion

Annualized cost of construction: \( \approx $30 \) billion ($560 billion @5%/year) (CBO, 1982)

Maintenance: \( \approx $70 \) billion (FHA, 2008)

Estimated annual gain of the IHS: \( \approx $150 - 200 \) billion

Welfare gain of IHS: 1.1 – 1.4%. Given homothetic preferences and holding prices fixed, can multiply welfare gain by U.S. GDP.

Suggests gains from IHS substantially greater than costs.