Today’s Plan

1. Overview
2. Log-supermodularity
3. Comparative advantage based assignment models
4. Cross-sectional predictions
5. Comparative static predictions
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Small but rapidly growing literature using assignment models in an international context:

- Trade: Grossman and Maggi (AER 2000); Grossman (JPE, 2004); Yeaple (JIE 2005); Ohnsorge and Trefler (JPE 2007); Costinot (Ecta 2009); Costinot and Vogel (JPE 2010); Sampson (2013); Grossman, Helpman and Kircher (2013)
- Offshoring: Kremer and Maskin (2003); Antras, Garicano and Rossi-Hansberg (QJE 2006); Nocke and Yeaple (REStud 2008); Costinot, Vogel and Wang (REStud 2013)

What do these models have in common?

- Factor allocation can be summarized by an assignment function
- Large number of factors and/or goods

What is the main difference between these models?

- Matching: Two sides of each match in finite supply (as in Becker 1973)
- Sorting: One side of each match in infinite supply (as in Roy 1951)
We restrict ourselves to sorting models, e.g. Ohnsorge and Trefler (2007), Costinot (2009), and Costinot and Vogel (2010)

- Production functions are linear, as in Ricardian model
- But more than one factor per country, as in Roy (1951) model
- “Ricardo-Roy model” (Costinot and Vogel, ARE 2015, survey)

Objectives:

1. Describe how these models relate to “standard” neoclassical models
2. Introduce simple tools from the mathematics of complementarity
3. Use tools to derive cross-sectional and comparative static predictions

NB: this will be a fairly methodological lecture (so for specific applications, see the papers).
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Log-supermodularity

Definition

- **Definition 1** A function $g : X \rightarrow \mathbb{R}^+$ is log-supermodular if for all $x, x' \in X$, $g(\max(x, x')) \cdot g(\min(x, x')) \geq g(x) \cdot g(x')$

- **Bivariate example:**
  - If $g : X_1 \times X_2 \rightarrow \mathbb{R}^+$ is log-spm, then $x'_1 \geq x'_1''$ and $x'_2 \geq x'_2''$ imply
    $$g(x'_1, x'_2) \cdot g(x'_1'', x'_2'') \geq g(x'_1, x'_2') \cdot g(x'_1'', x'_2,').$$
  - If $g$ is strictly positive, this can be rearranged as
    $$g(x'_1, x'_2) / g(x'_1'', x'_2') \geq g(x'_1, x'_2') / g(x'_1'', x'_2'').$$
Lemma 1. \( g, h : X \rightarrow \mathbb{R}^+ \) log-spm \( \Rightarrow \) \( gh \) log-spm

Lemma 2. \( g : X \rightarrow \mathbb{R}^+ \) log-spm \( \Rightarrow \) \( G(x_i) = \int_{X_i} g(x) \, dx \) log-spm

Lemma 3. \( g : T \times X \rightarrow \mathbb{R}^+ \) log-spm \( \Rightarrow \) 
\( x^*(t) \equiv \arg \max_{x \in X} g(t, x) \) increasing in \( t \)
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3. **Comparative advantage based assignment (R-R) models**
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Basic Environment

Consider a world economy with:

1. Multiple countries with characteristics $\gamma \in \Gamma$
2. Multiple goods or sectors with characteristics $\sigma \in \Sigma$
3. Multiple factors of production with characteristics $\omega \in \Omega$

Factors are immobile across countries, perfectly mobile across sectors

Goods are freely traded at world price $p(\sigma) > 0$
Within each sector, factors of production are perfect substitutes

\[ Q(\sigma, \gamma) = \int_{\Omega} A(\omega, \sigma, \gamma) L(\omega, \sigma, \gamma) d\omega, \]

\[ A(\omega, \sigma, \gamma) \geq 0 \] is productivity of \( \omega \)-factor in \( \sigma \)-sector and \( \gamma \)-country

**A1:** \( A(\omega, \sigma, \gamma) \) is log-supermodular

A1 implies, in particular, that:

1. High-\( \gamma \) countries have a comparative advantage in high-\( \sigma \) sectors
2. High-\( \omega \) factors have a comparative advantage in high-\( \sigma \) sectors
3.2.2. Firms. For future reference, it is useful to start by studying the cost-minimization problem of a representative firm, which is a necessary condition for profit maximization. By Equation 3, the unit-cost function of a firm producing good $g$, expressed as $c_g(s)$, is given by $c_g(s) = \frac{v_g(s) - d_{g,s}}{s}$. Having characterized the unit-cost function of a representative firm, its profit function can be written as $\pi = \sum_g (p_g - c_g(s)) q_g(s)$.

3.2.3. Market clearing. Factor and good market clearing finally require the set of factors, $V_R$, and factor allocation, $g$, such that Equations 3–9 hold. To summarize, a competitive equilibrium corresponds to consumption, $Z_d$, production, $Z_s$, and factor allocation, $g$, as given and explore how factor allocation, factor prices, and good prices, $p$, are determined.

- Can create arbitrary aggregate curvature from the right choice on distribution of micro heterogeneity
- Analogous to, e.g., results in demand theory described in Anderson, de Palma and Thisse (1992) or Berry and Haile (Ecta, 2014).

Figure 1
Production possibility frontiers in the Ricardo-Roy (R-R) model with two goods and $N = 1, 2, \infty$ factors.
V(ω, γ) ≥ 0 is inelastic supply of ω-factor in γ-country

A2: V(ω, γ) is log-supermodular

A2 implies that: high-γ countries are relatively more abundant in high-ω factors

Preferences will be described later on when we do comparative statics
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Competitive Equilibrium

- We take the price schedule $p(\sigma)$ as given (WLOG, since focusing on cross-section only)

- In a competitive equilibrium, $L$ and $w$ must be such that:
  1. Firms maximize profit

\[
p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) \leq 0, \text{ for all } \omega \in \Omega
\]
\[
p(\sigma) A(\omega, \sigma, \gamma) - w(\omega, \gamma) = 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0
\]

Note that the question of whether $L(\omega, \sigma, \gamma) > 0$ or not depends on $(\sigma, \omega, \gamma)$ but not also on some other $\sigma'$, $\omega'$, or $\gamma'$. Linearity of production function delivered this feature.

2. Factor markets clear

\[
V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma) \, d\sigma, \text{ for all } \omega \in \Omega
\]
Patterns of Specialization

Predictions

- Let $\Sigma (\omega, \gamma) \equiv \{\sigma \in \Sigma | L(\omega, \sigma, \gamma) > 0\}$ be the set of sectors in which factor $\omega$ is employed in country $\gamma$

- Theorem $\Sigma (\cdot, \cdot)$ is increasing

Proof:

1. Profit maximization $\Rightarrow \Sigma (\omega, \gamma) = \arg \max_{\sigma \in \Sigma} p(\sigma) A(\omega, \sigma, \gamma)$
2. $A_1 \Rightarrow p(\sigma) A(\omega, \sigma, \gamma)$ log-spm by Lemma 1
3. $p(\sigma) A(\omega, \sigma, \gamma)$ log-spm $\Rightarrow \Sigma (\cdot, \cdot)$ increasing by Lemma 3

- Corollary High-$\omega$ factors specialize in high-$\sigma$ sectors

- Corollary High-$\gamma$ countries specialize in high-$\sigma$ sectors
Ricardian model $\equiv$ Special case w/ $A(\omega, \sigma, \gamma) \equiv A(\sigma, \gamma)$

Previous corollary can help explain:

1. **Multi-country, multi-sector Ricardian model;** Jones (1961)
   - According to Jones (1961), efficient assignment of countries to goods solves $\max \sum \ln A(\sigma, \gamma)$
   - According to Corollary, $A(\sigma, \gamma)$ log-spm implies PAM of countries to goods; Becker (1973), Kremer (1993), Legros and Newman (1996).

   - Papers vary in terms of source of “institutional dependence” $\sigma$ and ”institutional quality” $\gamma$
   - ...but same fundamental objective: providing micro-theoretical foundations for the log-supermodularity of $A(\sigma, \gamma)$
Previous results are about the set of goods that each country produces.

**Question:** Can we say something about how much each country produces? Or how much it employs in each particular sector?

**Answer:** Without further assumptions, the answer is no.
A3. The profit-maximizing allocation $L$ is unique

A4. Factor productivity satisfies $A(\omega, \sigma, \gamma) \equiv A(\omega, \sigma)$

Comments:
1. A3 requires $p(\sigma)A(\omega, \sigma, \gamma)$ to be maximized in a single sector
2. A3 is an implicit restriction on the demand-side of the world-economy
   - ... but it becomes milder and milder as the number of factors or countries increases
   - ... generically true if continuum of factors
3. A4 implies no Ricardian sources of CA across countries
   - Pure Ricardian case (i.e. $A(\omega, \sigma, \gamma) \equiv A(\sigma, \gamma)$) can be studied in a similar fashion
   - Having multiple sources of CA is more complex (Costinot 2009)
Theorem If A3 and A4 hold, then $Q(\sigma, \gamma)$ is log-spm.

Proof:

1. Let $\Omega(\sigma) \equiv \{\omega \in \Omega | p(\sigma) A(\omega, \sigma) > \max_{\sigma' \neq \sigma} p(\sigma') A(\omega, \sigma')\}$. A3 and A4 imply $Q(\sigma, \gamma) = \int 1_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma) V(\omega, \gamma) d\omega$

2. $A1 \Rightarrow \tilde{A}(\omega, \sigma) \equiv 1_{\Omega(\sigma)}(\omega) \cdot A(\omega, \sigma)$ log-spm

3. $A2$ and $\tilde{A}(\omega, \sigma)$ log-spm + Lemma 1 $\Rightarrow \tilde{A}(\omega, \sigma)V(\omega, \gamma)$ log-spm

4. $\tilde{A}(\omega, \sigma)V(\omega, \gamma)$ log-spm + Lemma 2 $\Rightarrow Q(\sigma, \gamma)$ log-spm

Intuition:

1. $A1 \Rightarrow$ high $\omega$-factors are assigned to high $\sigma$-sectors

2. $A2 \Rightarrow$ high $\omega$-factors are more likely in high $\gamma$-countries

3. Compare to Rybczinski and H-O Theorems when you do H-O model with Kyle next quarter.
Corollary. Suppose that A3 and A4 hold. If two countries produce \( J \) goods, with \( \gamma_1 \geq \gamma_2 \) and \( \sigma_1 \geq \ldots \geq \sigma_J \), then the high-\( \gamma \) country tends to specialize in the high-\( \sigma \) sectors:

\[
\frac{Q(\sigma_1, \gamma_1)}{Q(\sigma_1, \gamma_2)} \geq \ldots \geq \frac{Q(\sigma_J, \gamma_1)}{Q(\sigma_J, \gamma_2)}
\]
Aggregate Output, Revenues, and Employment

Employment and revenue predictions

Let $L(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} V(\omega, \gamma) d\omega$ be aggregate employment.

Let $R(\sigma, \gamma) \equiv \int_{\Omega(\sigma)} r(\omega, \sigma) V(\omega, \gamma) d\omega$ be aggregate revenues.

**Corollary.** Suppose that A3 and A4 hold. If two countries produce J goods, with $\gamma_1 \geq \gamma_2$ and $\sigma_1 \geq ... \geq \sigma_J$, then aggregate employment and aggregate revenues follow the same pattern as aggregate output:

$$\frac{L(\sigma_1, \gamma_1)}{L(\sigma_1, \gamma_2)} \geq ... \geq \frac{L(\sigma_J, \gamma_1)}{L(\sigma_J, \gamma_2)} \quad \text{and} \quad \frac{R(\sigma_1, \gamma_1)}{R(\sigma_1, \gamma_2)} \geq ... \geq \frac{R(\sigma_J, \gamma_1)}{R(\sigma_J, \gamma_2)}$$
Aggregate Output, Revenues, and Employment

Relation to the previous literature

1. **Worker Heterogeneity and Trade**
   - Generalization of Ruffin (1988):
     - Continuum of factors, Hicks-neutral technological differences
     - Results hold for an arbitrarily large number of goods and factors
   - Generalization of Ohnsorge and Trefler (2007):
     - No functional form assumption (log-normal distribution of human capital, exponential factor productivity)

2. **Firm Heterogeneity and Trade**
     - “Factors” ≡ “Firms” with productivity $\omega$
     - “Countries” ≡ “Industries” with characteristic $\gamma$
     - “Sectors” ≡ “Organizations” with characteristic $\sigma$
     - $Q(\sigma, \gamma)$ ≡ Sales by firms with ”$\sigma$-organization” in “$\gamma$-industry”
   - In previous papers, $f(\omega, \gamma)$ log-spmm is crucial, Pareto is not. (For more on this, see w.p. version of Costinot, 2009.)
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Closing The Model

Additional assumptions

- Most of this will follow Costinot and Vogel (JPE, 2010)
- Assumptions A1-A4 are maintained
- In order to do comparative statics (unless we assume a "small open economy" that takes all goods prices as given), we also need to specify the demand side of our model:

\[
U = \left\{ \int_{\sigma \in \Sigma} \left[ C(\sigma, \gamma) \right]^{\frac{\varepsilon-1}{\varepsilon}} d\sigma \right\}^{\frac{\varepsilon}{\varepsilon-1}}
\]

- For expositional purposes (but see wp version of CV 2010) for discrete case, we will also assume that:
  - \( A(\omega, \sigma) \) is strictly log-supermodular
  - Continuum of factors and sectors: \( \Sigma \equiv [\underline{\sigma}, \overline{\sigma}] \) and \( \Omega \equiv [\underline{\omega}, \overline{\omega}] \)
Autarky equilibrium is a set of functions \((Q, C, L, p, w)\) such that:

1. **Firms maximize profits**

   \[
   p(\sigma)A(\omega, \sigma) - w(\omega, \gamma) \leq 0, \text{ for all } \omega \in \Omega \\
   p(\sigma)A(\omega, \sigma) - w(\omega, \gamma) = 0, \text{ for all } \omega \in \Omega \text{ s.t. } L(\omega, \sigma, \gamma) > 0
   \]

2. **Factor markets clear**

   \[
   V(\omega, \gamma) = \int_{\sigma \in \Sigma} L(\omega, \sigma, \gamma) \, d\sigma, \text{ for all } \omega \in \Omega
   \]

3. **Consumers maximize their utility and goods markets clear**

   \[
   C(\sigma, \gamma) = I(\gamma) \times p(\sigma)^{-\varepsilon} = Q(\sigma, \gamma)
   \]
Lemma In autarky equilibrium, there exists an increasing bijection $M : \Omega \rightarrow \Sigma$ such that $L(\omega, \sigma, \gamma) > 0$ if and only if $M(\omega, \gamma) = \sigma$.

Lemma In autarky equilibrium, $M$ and $w$ satisfy

$$\frac{dM(\omega, \gamma)}{d\omega} = \frac{A[\omega, M(\omega, \gamma)] V(\omega, \gamma)}{I(\gamma) \times \{p[M(\omega), \gamma]\}^{-\varepsilon}}$$

(1)

$$\frac{d\ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega)]}{\partial \omega}$$

(2)

with $M(\omega, \gamma) = \sigma$, $M(\overline{\omega}, \gamma) = \overline{\sigma}$, and $p[M(\omega, \gamma), \gamma] = w(\omega, \gamma) / A[\omega, M(\omega, \gamma)]$. 

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Question: What happens if we change country characteristics from $\gamma$ to $\gamma' \leq \gamma$?

If $\omega$ is worker “skill”, this can be thought of as a change in terms of “skill abundance”:

$$\frac{V(\omega, \gamma)}{V(\omega', \gamma)} \geq \frac{V'(\omega, \gamma')}{V'(\omega', \gamma')}, \text{ for all } \omega > \omega'$$

If $V(\omega, \gamma)$ was a normal distribution, this would correspond to a change in the mean.
Changes in Factor Supply

Consequence for factor allocation

- **Lemma** $M(\omega, \gamma') \geq M(\omega, \gamma)$ for all $\omega \in \Omega$

- **Intuition:**
  - If there are relatively more low-$\omega$ factors, more sectors should use them
  - From a sector standpoint, this requires *factor downgrading*
Proof: If there is $\omega$ s.t. $M(\omega, \gamma') < M(\omega, \gamma)$, then there exist:

1. $M(\omega_1, \gamma') = M(\omega_1, \gamma) = \sigma_1$, $M(\omega_2, \gamma') = M(\omega_2, \gamma) = \sigma_2$, and $\frac{M_{\omega}(\omega_1, \gamma')}{M_{\omega}(\omega_2, \gamma')} \leq \frac{M_{\omega}(\omega_1, \gamma)}{M_{\omega}(\omega_2, \gamma)}$.

2. Equation (1) \[ \frac{V(\omega_2, \gamma')}{V(\omega_1, \gamma')} \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} \geq \frac{V(\omega_2, \gamma)}{V(\omega_1, \gamma)} \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)} \]

3. $V \text{ log-spm} \implies \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} \geq \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)}$.

4. Equation (2) + zero profits \[ \frac{d \ln p(\sigma, \gamma)}{d\sigma} = - \frac{d \ln A[M^{-1}(\sigma, \gamma), \sigma]}{d\sigma} \]

5. $M^{-1}(\sigma, \gamma) < M^{-1}(\sigma, \gamma')$ for $\sigma \in (\sigma_1, \sigma_2) + A \text{ log-spm}$ \[ \frac{p(\sigma_1, \gamma)}{p(\sigma_2, \gamma)} < \frac{p(\sigma_1, \gamma')}{p'(\sigma_2, \gamma')} \]

6. $\frac{p(\sigma_1, \gamma)}{p(\sigma_2, \gamma)} < \frac{p(\sigma_1, \gamma')}{p'(\sigma_2, \gamma')}$ + CES \[ \frac{C(\sigma_1, \gamma')}{C(\sigma_2, \gamma')} > \frac{C(\sigma_1, \gamma)}{C(\sigma_2, \gamma)} \] A contradiction.
Changes in Factor Supply
Consequence for factor prices

- A decrease from $\gamma$ to $\gamma'$ implies *pervasive rise in inequality*:

$$\frac{w(\omega, \gamma')}{w(\omega', \gamma')} \geq \frac{w(\omega, \gamma)}{w(\omega', \gamma)} \text{, for all } \omega > \omega'$$

- The mechanism is simple:

  1. Profit-maximization implies

$$\frac{d \ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma)]}{\partial \omega}$$

$$\frac{d \ln w(\omega, \gamma')}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma')]}{\partial \omega}$$

  2. Since $A$ is log-supermodular, task upgrading implies

$$\frac{d \ln w(\omega, \gamma')}{d\omega} \geq \frac{d \ln w(\omega, \gamma)}{d\omega}$$
Costinot and Vogel (2010) also consider changes in diversity. This corresponds to the case where there exists \( \hat{\omega} \) such that \( V(\omega, \gamma) \) is log-supermodular for \( \omega > \hat{\omega} \), but log-submodular for \( \omega < \hat{\omega} \).

Also consider changes in factor demand (Computerization?):

\[
U = \left\{ \int_{\sigma \in \Sigma} B(\sigma, \gamma) \left[ C(\sigma, \gamma) \right]^\frac{\epsilon-1}{\epsilon} d\sigma \right\} \frac{\epsilon}{\epsilon-1}
\]
North-South Trade
Free trade equilibrium

- Two countries, Home ($H$) and Foreign ($F$), with $\gamma_H \geq \gamma_F$
- A competitive equilibrium in the world economy under free trade requires that there exists some $\gamma_T$ s.t.:

\[
\frac{dM(\omega, \gamma_T)}{d\omega} = \frac{A[\omega, M(\omega, \gamma_T)] V(\omega, \gamma_T)}{l_T \times \{p[M(\omega, \gamma_T), \gamma_T]\}^{-\varepsilon}},
\]

\[
\frac{d \ln w(\omega, \gamma_T)}{d\omega} = \frac{\partial \ln A[\omega, M(\omega, \gamma_T)]}{\partial \omega},
\]

where:

\[
M(\omega, \gamma_T) = \sigma \text{ and } M(\overline{\omega}, \gamma_T) = \overline{\sigma}
\]

\[
p[M(\omega, \gamma_T), \gamma_T] = w(\omega, \gamma_T) A[\omega, M(\omega, \gamma_T)]
\]

\[
V(\omega, \gamma_T) \equiv V(\omega, \gamma_H) + V(\omega, \gamma_F)
\]
Key observation:
\[
\frac{V(\omega, \gamma_H)}{V(\omega', \gamma_H)} \geq \frac{V(\omega, \gamma_F)}{V(\omega', \gamma_F)}, \text{ for all } \omega > \omega' \implies \frac{V(\omega, \gamma_H)}{V(\omega', \gamma_H)} \geq \frac{V(\omega, \gamma_T)}{V(\omega', \gamma_T)} \geq \frac{V(\omega, \gamma_F)}{V(\omega', \gamma_F)}
\]

This then means that we can apply our autarky comparative statics (on change in $\gamma$) from above since moving from, say, $H$ in autarky to free trade is just like moving to an equilibrium with a lower $\gamma$.

Continuum-by-continuum extensions of two-by-two HO results:

1. Changes in skill-intensities:
\[
M(\omega, \gamma_H) \leq M(\omega, \gamma_T) \leq M(\omega, \gamma_F), \text{ for all } \omega
\]

2. Strong Stolper-Samuelson effect:
\[
\frac{w(\omega, \gamma_H)}{w(\omega', \gamma_H)} \leq \frac{w(\omega, \gamma_T)}{w(\omega', \gamma_T)} \leq \frac{w(\omega, \gamma_F)}{w(\omega', \gamma_F)}, \text{ for all } \omega > \omega'
\]
North-South trade driven by factor demand differences:
- Same logic gets to the exact opposite results
- Correlation between factor demand and factor supply considerations matters

One can also extend analysis to study “North-North” trade:
- It predicts wage polarization in the more diverse country and wage convergence in the other
What’s next/Other work?

- Recent survey by Costinot and Vogel (ARE 2015)
- Extensions:
  - Beyond perfect competition (but see Sampson, 2013a, for monopolistic competition; Acemoglu-Jensen for aggregative games)
  - Sorting and matching (see e.g. Grossman, Helpman and Kircher, 2014)
  - Search frictions (e.g. tools in Shimer and Smith)
  - Tools from wider matching (and optimal transport) literature.
- Dynamic issues:
  - Sector-specific human capital accumulation
  - Endogenous and directed technology adoption/innovation
- Empirics:
  - Revisiting the consequences of trade liberalization more flexibly
  - Testing comparative advantage more flexibly
  - Patterns of specialization across cities (Davis and Dingel, 2014)
  - Tools from the Labor literature? (Heckman-Sedlacek, Heckman-Scheinkman, Heckman-Honore, Sattinger)
  - Explaining wage patterns by sectors (see Sampson, 2013b)